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A PARAMETRIC ANALYSIS FOR SINGLE MACHINE SCHEDULING WITH PAST-SEQUENCE-DEPENDENT SETUP TIMES

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Abstract. *Recently, the single machine scheduling problem with past-sequence-dependent (p-s-d) setup times is getting more attentions from academic researchers and industrial practitioners. The past-sequence-dependent setup times are proportional to the length of already scheduled jobs. It is shown that for a number of objective functions this scheduling problem can be solved in $O(n \log n)$ time. In this paper, we extend the analysis of the problem with the total absolute difference in completion times (TADC) as the objective function. This problem is denoted as $1/s_{psd}/TADC$ in [1]. Let $s_{[j]}$ and $p_{[j]}$ be the setup time and processing time of a job occupying position j in the sequence respectively, and $s_{[j]}$ is defined as $s_{[j]} = \gamma \sum_{i=1}^{j-1} p_{[i]}$, where γ is a normalizing constant. In this paper, we present a parametric analysis of γ on the $1/s_{psd}/TADC$ problem. We show analytically the number of optimal sequences and the range of γ in which each of the sequence is optimal. We prove that the number of optimal sequences is $\{1 + \sum_{k=1}^{x}(2k)\}$ if n is odd, and $\{1+\sum_{k=1}^{x}(2k-1)\}$ if n is even. The value of x is $\lfloor \frac{n}{2} \rfloor - 1$ when n is odd, and x is $\frac{n}{2}$ when is even. The number of optimal sequences depends only on n, the number of jobs, and not on γ. We also show analytically that when $\gamma > \frac{(n-3)}{2(n-2)}$, the optimal sequence is unique and is obtained by placing the longest job in first position and the rest of the jobs in SPT order in positions 2 to n.*

1. **Introduction.** In a recent study [1], the concept of past-sequence-dependent setup times is introduced in the well-known single machine scheduling problem. In their study, the setup time is dependent on the jobs that are already scheduled. The objectives considered in their study are minimizing maximum completion time (C_{max}), total completion time (TC), total absolute difference in completion times ($TADC$), and a bi-criterion objective function with TC and $TADC$. It is shown in their study that the single machine scheduling problem (with past-sequence-dependent setup times) with the above objectives can be solved in $O(n \log n)$ time. The learning effect has been included by [2] in the single machine scheduling problem with past-sequence-dependent setup times. In this study [2], a set of objectives including C_{max}, TC, $TADC$, along with minimizing the sum of earliness, tardiness and common due date ($ETCP$) penalties is considered. Polynomial time algorithms are proposed by [2] to obtain the optimal solution for the

objective functions considered. An extension of single machine scheduling problem with past-sequence-dependent setup times with due dates is presented by [3]. It is shown by [3] that minimizing total lateness, total tardiness (with agreeable due dates), maximum lateness (with agreeable due dates), maximum tardiness (with agreeable due dates) are solvable in polynomial time. The algorithm given by [3] does not guarantee optimal solution when minimizing the number of tardy jobs, maximum lateness and maximum tardiness are considered. References [4] and [5] presents a survey of scheduling with startup times. But in these surveys they do not consider the past-sequence-dependent setup times. A parametric analysis provided by [6] for a given learning index α was presented to show the range of δ in which a sequence is optimal. The parametric approach is adopted in this research in dealing with this single machine scheduling with setup times. To the best of our knowledge, reference [1] is the first in the literature to introduce the concept of past-sequence-dependent setup times in single-machine scheduling problems.

In this paper, we consider the non-preemptive single machine scheduling problem. A batch of n independent jobs to be processed on a continuously available single machine. The machine can process only one job at a time and job splitting and inserting idle times are not permitted. All the jobs are available at time zero. Each job has a processing time p_j, $(j = 1, 2, ..., n)$. Let $s_{[j]}$ and $p_{[j]}$ be the setup time and processing time of a job occupying position j in the sequence respectively, and $s_{[j]}$ is defined as

$$s_{[j]} = \gamma \sum_{i=1}^{j-1} p_{[i]} \; j = 2, 3, ..., n \; s_{[1]} = 0, \tag{1}$$

where $\gamma \geq 0$ is a normalizing constant. In the above Eq.(1), the actual length of the setup time depends on the value of γ. Reference [1] considered the following scheduling problems with past-sequence-dependent setup times given by equation (1). Problem.(i): $1/s_{psd}/C_{max}$; Problem.(ii):$1/s_{psd}/TC$; Problem.(iii):$1/s_{psd}/TADC$; Problem.(iv):$1/s_{psd}/BC$. It is shown in [1] that the well-known shortest processing time (SPT) sequence is optimal for both the problems Problem. (i) $(1/s_{psd}/C_{max})$ and Problem.(ii) $(1/s_{psd}/TC)$.

Contributions of this paper: We consider the problem $(1/s_{psd}/TADC)$. For this problem, the optimal sequence depends on the value of γ. We present a parametric analysis of γ on the $1/s_{psd}/TADC$ problem. We show analytically the number of optimal sequences and the range of γ in which each of the sequence is optimal. It is shown in this paper that the number of optimal sequences is $\{1 + \sum_{k=1}^{x}(2k)\}$ if n is odd, and $\{1 + \sum_{k=1}^{x}(2k - 1)\}$ if n is even. The value of x is $\lfloor \frac{n}{2} \rfloor - 1$ when n is odd, and x is $\frac{n}{2}$ when is even. The number of optimal sequences depends only on n the number of jobs and not on γ. We also show analytically that when $\gamma > \frac{(n-3)}{2(n-2)}$, the optimal sequence is unique and is obtained by placing the longest job in first position and the rest of the jobs in SPT order in positions 2 to n.

In terms of the contribution for the industry, it is indicated by [1] that the consideration of past-sequence-dependent setup times stems from high-tech manufacturing in which a batch of jobs consists of a group of electronic components mounted together on an IC board. References [7,8] also indicate the importance of scheduling problem in considering other constraints such as transportation routing or container transfer scheduling. It is mentioned in reference [9] that effects of Inventory Control on Bullwhip in production planning and scheduling are very significant for manufacturing companies. In addition, reference [10] mentioned more general manufacturing environment in which either long setup times are common. As a result, the P-S-D problem is very important and practical to be taken into account in most of the manufacturing factory as they are very often encountered on the shop floor.

In Section 2, we formally define the problem and present a motivating numerical example for understanding the contributions of our study. In Section 3, we consider a generalized problem and present some interesting analytical results. The conclusions of this study is presented in Section 4.

2. **Problem Definition $1/s_{psd}/TADC$.** In this section, we consider the single-machine scheduling problem with the objective of minimizing the total absolute difference in completion times (TADC). This problem is denoted as $1/s_{psd}/TADC$ in the standard notation used in literature. Reference [11] was the first to show that

$$\begin{aligned} TADC &= \sum_{i=1}^{n}\sum_{j=i}^{n}|C_j - C_i| \\ &= \sum_{r=1}^{n}(n-r)(n-r+1)\ (s_{[r]} + p_{[r]}) \\ &= \sum_{r=1}^{n}\left\{(r-1)(n-r+1) + \gamma \sum_{j=r+1}^{n}(j-1)(n-j+1)\right\} p_{[r]} \end{aligned} \tag{2}$$

As mentioned in [1], Eq.(2) can be viewed as scalar product of two vectors. One vector is $p_{[r]}$ the vector of processing time of jobs. The other vector is v_r known as positional weights vector given as

$$v_r = (r-1)(n-r+1) + \gamma \sum_{j=r+1}^{n}(j-1)(n-j+1),\ r = 2, 3, ..., n \tag{3}$$

In the above Eq.(3), the value of $v_1 = 0$ because $s_{[1]}$ is zero. It is well-known from [12] that Eq.(2) is minimized by arranging the elements of one vector in in non-increasing order and the elements of other vector in nondecreasing order. This is known as HLP Theorem. Hence, for a given value of γ, using HLP theorem, the optimal sequence for the $1/s_{psd}/TADC$ problem can be obtained in $O(n\ log\ n)$ time. It can be seen that the optimal sequence depends on the value of γ.

Motivating Numerical Example: Let us consider the 7 job example given in [12]. The processing time of jobs are: $p_1 = 2$, $p_2 = 3$, $p_3 = 6$, $p_4 = 9$, $p_5 = 21$, $p_6 = 65$ and $p_7 = 82$. Let us consider the value of $\gamma = 0.5$.

For a given value of $\gamma = 0.5$, the positional weights (v_r) are obtained from Eq.(3) as: $v_1 = 0$, $v_2 = 31$, $v_3 = 30$, $v_4 = 26$, $v_5 = 20$, $v_6 = 13$, and $v_7 = 6$. Using HLP theorem, the optimal sequence is obtained. It is shown in [1], the optimal sequence is {7 1 2 3 4 5 6} and the value of $TADC$ for this sequence is 1151.

Proposition.1: We multiply (or increase) all the elements of vector v_r obtained from Eq.(3), by a factor q ($q > 0$) and call this vector v_r^q. The optimal sequence obtained (using HLP theorem) with this vector v_r^q, will be the same optimal sequence we obtained (using HLP theorem) with the vector v_r.

Parametric analysis of γ: We know that the optimal sequence for the $1/s_{psd}/TADC$ problem depends on the value of γ. Our interest is to find the range of γ and the corresponding optimal sequence. For this purpose, we plot the value of v_r, ($r = 1, 2, ..., n$) with the value of γ. For this 7 job numerical example given by [11], the positional weight vector (v_r) is given by $v_1 = 0$, $v_2 = 6 + 50\gamma$, $v_3 = 10 + 40\gamma$, $v_4 = 12 + 28\gamma$, $v_5 = 12 + 16\gamma$, $v_6 = 10 + 6\gamma$, and $v_7 = 6$. The variation of v_r, ($r = 1, 2, ..., n$) for γ values in the range (0, 0.5) is shown in Figure.1. The variation of v_r with γ are linear and so we call them as lines $v_1, v_2, ..., v_7$. We see that lines $v_1 = 0$ and $v_7 = 6$ are independent of γ. We also see that v_1 and v_2 are less than v_3, v_4, v_5 and v_6 for $\gamma > 0$. In Figure.1, we see that there

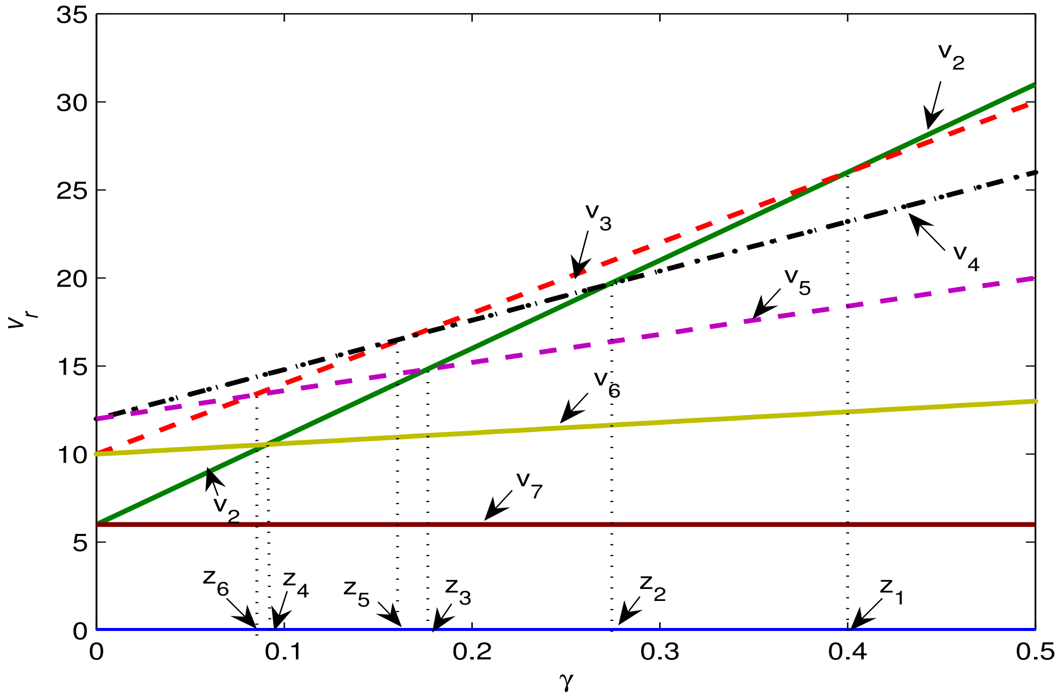

FIGURE 1. Variation of v_r as a function of γ

is a range of γ in which the lines v_r, $(r = 2, 3, ..., 6)$ will not intersect each other. (See for example the range of $\gamma = 0.30$ and $\gamma = 0.35$). This implies that there is a unique optimal sequence in that range of γ. In this range, all the v_r values $r = 2, 3, ..., 6$ are increasing with γ. From Proposition.1, we say that the optimal sequence in this range is unique.

Hence, in order to obtain the range of γ in which a sequence is optimal, we have to obtain the intersection points of all lines v_r, $(r = 2, 3, ..., 6)$ for $\gamma > 0$. These intersection points can be obtained by equating the corresponding positional weights (v_r). For example, the intersection point of lines v_2 and v_3 is obtained as: $v_2 = v_3$ which implies $(6 + 50\gamma) = (10+40\gamma)$. From this we get $10\gamma = 4$ and hence $\gamma = \frac{4}{10}$. For this 7 job numerical example, there are six points of intersection denoted as z_1 to z_6. These intersection points can also be seen from the Figure.1. These intersection points are: Lines v_2 and v_3 will intersect at point $z_1 = \frac{4}{10}$. Lines v_2 and v_4 will intersect at point $z_2 = \frac{6}{22}$. Lines v_2 and v_5 will intersect at point $z_3 = \frac{6}{34}$. Lines v_2 and v_6 will intersect at point $z_4 = \frac{4}{44}$. Lines v_3 and v_4 will intersect at point $z_5 = \frac{2}{12}$. v_3 and v_5 will intersect at point $z_6 = \frac{2}{24}$. We arrange these 6 intersection points z_1 to z_6 in the increasing order given by $\{z_6\ z_4\ z_5\ z_3\ z_2\ z_1\}$. We choose a value γ in between any two consecutive values of z (say between z_4 and z_5) and obtain the optimal sequence using the HLP Theorem. This sequence is optimal in the range of γ given by z values (z_4 and z_5). In this manner, we obtain 7 optimal sequences. The optimal sequences and the range of γ are shown in Table.1. Note that, we have to use one value of $0 < \gamma < z_6$ and one value of $\gamma > z_1$ and obtain the corresponding optimal sequences.

From the above numerical example, we observe the following: The longest job (job number 7) will always occupy the first position in the optimal sequence (because $v_1 = 0$). The second longest job will always occupy the last position in the optimal sequence (because $v_6 < v_2, v_3, v_4, v_5$ for $\gamma > 0$). The number of intersection points is 6. The

TABLE 1. Range of γ and the optimal sequence for $1/s_{psd}/TADC$ problem

Range of γ	Optimal Sequence
0.0 to $\frac{2}{24}$	$\{7,5,3,1,2,4,6\}$
$\frac{2}{24}$ to $\frac{4}{44}$	$\{7,5,2,1,3,4,6\}$
$\frac{4}{44}$ to $\frac{2}{12}$	$\{7,4,2,1,3,5,6\}$
$\frac{2}{12}$ to $\frac{6}{34}$	$\{7,4,1,2,3,5,6\}$
$\frac{6}{34}$ to $\frac{6}{22}$	$\{7,3,1,2,4,5,6\}$
$\frac{6}{22}$ to $\frac{4}{10}$	$\{7,2,1,3,4,5,6\}$
Greater than $\frac{4}{10}$	$\{7,1,2,3,4,5,6\}$

number of optimal sequences is number of intersections plus one i.e., 7. This because we have to include the value of γ for $0 < \gamma < z_6$ and $\gamma > z_1$.

At any point of intersection there are two sequences that are optimal. For example, when $\gamma = 0.4$ both the sequences $\{7,2,1,3,4,5,6\}$ and $\{7,1,2,3,4,5,6\}$ are optimal, which implies that the value of $TADC$ is same and is 1085.2. For the value of $\gamma > \frac{4}{10}$, there are no intersections of the lines. This implies that when $\gamma > \frac{4}{10}$, the optimal sequence is unique and is $\{7,1,2,3,4,5,6\}$.

3. **Generalization to n Jobs.**

In this section, we will generalize the results for any value of n. For this purpose, we first consider n is an odd number, and then consider the case when n is even.

The number of jobs (n) is odd: We know that $v_1 = 0$ and $v_2.v_3, ..., v_n$ are given by Eq.(3). When n is odd, line v_2 will intersect with lines $v_3,\ v_4,\ ..., v_{n-1}$ for $\gamma > 0$. Line v_2 will intersect with v_n at $\gamma = 0$. Thus, there are $(n-3)$ distinct intersection points $(\gamma_k > 0)$. The value of γ_k at these $(n-3)$ intersection points are given by

$$\gamma_k = \frac{k(n-k)-(n-1)}{\sum_{l=2}^{k} l(n-l)},\ k = 2,3,...,(n-2) \tag{4}$$

A detailed proof of the above Eq.(4) is given in Appendix.

Similarly, line v_3 will intersect with lines $v_4,\ v_5,\ ..., v_{n-2}$ for $\gamma > 0$. Line v_3 will intersect with lines v_{n-1}, v_n at $\gamma \leq 0$. Thus, there are $(n-5)$ distinct intersection points $(\gamma_k > 0)$. The value of γ_k at these $(n-5)$ intersection points are given by

$$\frac{k(n-k)-2(n-1)}{\sum_{l=3}^{k} l(n-l)},\ k = 3,4,...,(n-3) \tag{5}$$

In the same manner, v_4 will have $(n-7)$ intersection points (γ_k values) given by

$$\frac{k(n-k)-3(n-1)}{\sum_{l=4}^{k} l(n-l)},\ k = 4,5,...,(n-4) \tag{6}$$

These above Eqs. (5) and (6) can be easily obtained by following the steps given for Eq.(4) in the Appendix.

We now define $x = \lfloor \frac{n}{2} \rfloor - 1$. The line v_x will intersect with lines v_{x+1} and v_{x+2} for $\gamma > 0$. Thus, there are two intersection points. The line v_x will intersect with lines $v_{x+3}, v_{x+4}, ..., v_n$ for $\gamma \leq 0$.

Lines $v_{x+1}, v_{x+2}, ..., v_{n-1}$ will not intersect with each other for $\gamma > 0$.

Hence, in general (when n is odd), the total number of intersection points are

$$(n-3) + (n-5) + (n-7) + (n-9) + ... + 2 \tag{7}$$

The above Eq.(7) is the sum of even number (i.e., $2+4+6+...$). So the total number of intersection points is the addition of even numbers up to x terms given by

$$\sum_{k=1}^{x} 2k \tag{8}$$

We know the total number of intersection points. Here, we call these intersection points as distinct values of γ. We arrange these distinct values γ in increasing order and obtain the optimal sequence for each range in between these distinct values γ (as done in the example). In this manner, we obtain $\{1 + \sum_{k=1}^{x} 2k\}$ optimal sequences and the range of γ in which each of these sequence is optimal.

The number of optimal sequences is number of intersections plus one. This is because we have to consider $0 < \gamma < \gamma_{min}$ and $\gamma > \gamma_{max}$, where γ_{min} and γ_{max} are the minimum and maximum values of distinct values of γ respectively. When $n = 7$, we get $x = 2$. We get from Eq.(4) and Eq.(5) 4 and 2 intersection points respectively. So we obtain the number of optimal sequences as 7.

The number of jobs (n) is even: When n is even, line v_2 will intersect with lines $v_3,\ v_4,\ ...,v_{n-1}$ for $\gamma > 0$. Thus, there are $(n-3)$ intersection points given by Eq.(4). Line v_3 will intersect with lines $v_4,\ v_5,\ ...,v_{n-2}$ for $\gamma > 0$. Thus, there are $(n-5)$ intersection points given by Eq. (5). In the same manner, v_4 will have $(n-7)$ intersection points given by Eq.(6).

We now define $x = \frac{n}{2}$. The line v_x will intersect with line v_{x+1} for $\gamma > 0$. Thus, there is only one intersection point. The line v_x will intersect with lines $v_{x+2}, v_{x+3}, ..., v_n$ for $\gamma \leq 0$.

Lines $v_{x+1}, v_{x+2}, ..., v_{n-1}$ will not intersect with each other for $\gamma > 0$.

Hence, in general (when n is even), the total number of intersection points are

$$(n-3) + (n-5) + (n-7) + (n-9) + ... + 1 \tag{9}$$

When n is even, the above Eq.(9) is the sum of odd numbers (i.e., $1+3+5+...$). The total number of distinct intersection points is the addition of odd numbers up to x terms and is given by

$$\sum_{k=1}^{x} (2k-1) \tag{10}$$

The number of optimal sequences is number of intersections plus one. This because in each interval we have one sequence that is optimal. Hence, the number of optimal sequences is $\{1 + \sum_{k=1}^{x}(2k-1)\}$.

When n is even, We know the total number of intersection points; i.e., we know the distinct values of γ. We arrange these γ values in increasing order (include $\gamma = 0$). We choose a γ value (in between two γ values given above) and obtain the sequence using the HLP Theorem. In this manner, we obtain $\{1 + \sum_{k=1}^{x}(2k-1)\}$ optimal sequences.

We also see from the values of v_r that the maximum value of γ is given by the point of intersection of v_2 and v_3. This maximum value of γ is denoted as γ_{Max} and is

$$\gamma_{Max} = \frac{n-3}{2(n-2)} \tag{11}$$

Beyond this above value of γ_{Max}, there will not be any intersection. This implies that when $\gamma > \frac{n-3}{2(n-2)}$, the optimal sequence is unique and is obtained by placing the longest job in first position and the rest of the jobs in SPT order in positions 2 to n. This is true for both when n is odd and even.

4. **Conclusions.** We considered the single machine scheduling problem with past-sequence-dependent setup times that are proportional to the length of already scheduled jobs, that is, with past-sequence-dependent (p-s-d) setup times. We presented a parametric analysis (of γ) for the problem with the total absolute difference in completion times (TADC) as the objective function denoted as $1/s_{psd}/TADC$ in [1]. We have shown analytically the set of optimal sequences and the range of γ in which each of the sequences are optimal. We have proved that the number of optimal sequences is $\{1 + \sum_{k=1}^{x}(2k)\}$ if n is odd, and $\{1 + \sum_{k=1}^{x}(2k-1)\}$ if n is even. The value of x is $\lfloor \frac{n}{2} \rfloor - 1$ when n is odd and x is $\frac{n}{2}$ when is even. We have also shown analytically that when $\gamma > \frac{(n-3)}{2(n-2)}$, the optimal sequence is unique and is obtained by placing the longest job in first position and the rest of the jobs in SPT order in positions 2 to n.

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Appendix. Here, we will explain how we obtained the Eq.(4) in a detailed manner. We first write the vector v_r for a general n as

$$
\begin{aligned}
v_1 &= 0 && (12)\\
v_2 &= 1*(n-1) \;+\; \gamma\sum_{j=3}^{n}(j-1)(n-j+1) && (13)\\
v_3 &= 2*(n-2) \;+\; \gamma\sum_{j=4}^{n}(j-1)(n-j+1) && (14)\\
v_4 &= 3*(n-3) \;+\; \gamma\sum_{j=5}^{n}(j-1)(n-j+1) && (15)\\
\ldots. &\;\;.. \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &&\\
v_{n-2} &= (n-3)*3 \;+\; \gamma\sum_{j=(n-1)}^{n}(j-1)(n-j+1) && (16)\\
v_{n-1} &= (n-2)*2 \;+\; \gamma\sum_{j=n}^{n}(j-1)(n-j+1) && (17)\\
v_n &= (n-1)*1 && (18)
\end{aligned}
$$

Equating v_2 and v_3, we get

$$1*(n-1) \;+\; \gamma\sum_{j=3}^{n}(j-1)(n-j+1) \;=\; 2*(n-2) \;+\; \gamma\sum_{j=4}^{n}(j-1)(n-j+1) \quad (19)$$

This above equation can be rewritten as

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=4}^{n}(j-1)(n-j+1)\right\}\right] \;=\; 2*(n-2)-1*(n-1) \quad (20)$$

It can be seen that

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=4}^{n}(j-1)(n-j+1)\right\}\right] \;=\; \gamma[2*(n-2)] \quad (21)$$

Hence, (20) reduces to

$$\gamma[2*(n-2)] \;=\; 2*(n-2)-1*(n-1) \quad (22)$$

From the above we get

$$\gamma \;=\; \frac{2*(n-2)-1*(n-1)}{[2*(n-2)]} \quad (23)$$

This above equation is the same with $k=2$ in Eq.(4).

We obtain the following equations, when we equate v_2 and v_4:

$$1*(n-1) \;+\; \gamma\sum_{j=3}^{n}(j-1)(n-j+1) \;=\; 3*(n-3) \;+\; \gamma\sum_{j=5}^{n}(j-1)(n-j+1) \quad (24)$$

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=5}^{n}(j-1)(n-j+1)\right\}\right] \;=\; 3*(n-3)-1*(n-1) \quad (25)$$

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=5}^{n}(j-1)(n-j+1)\right\}\right] \;=\; \gamma\left[\sum_{j=3}^{4}(j-1)(n-j+1)\right] \quad (26)$$

$$\gamma\left[\sum_{j=3}^{4}(j-1)(n-j+1)\right] \;=\; 3*(n-3)-1*(n-1) \tag{27}$$

From the above we get

$$\gamma \;=\; \frac{3*(n-3)-1*(n-1)}{[\sum_{j=3}^{4}(j-1)(n-j+1)]} \;=\; \frac{3*(n-3)-1*(n-1)}{2*(n-2)+3*(n-3)} \tag{28}$$

This above equation is the same with $k = 3$ in Eq. (4).

We obtain the following equations, when we equate v_2 and v_{n-1}:

$$1*(n-1) \;+\; \gamma\sum_{j=3}^{n}(j-1)(n-j+1) \;=\; (n-2)*2 \;+\; \gamma\sum_{j=n}^{n}(j-1)(n-j+1) \tag{29}$$

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=n}^{n}(j-1)(n-j+1)\right\}\right] \;=\; (n-2)*2-1*(n-1) \tag{30}$$

$$\gamma\left[\left\{\sum_{j=3}^{n}(j-1)(n-j+1)\right\}-\left\{\sum_{j=n}^{n}(j-1)(n-j+1)\right\}\right] \;=\; \gamma\left[\sum_{j=3}^{n-1}(j-1)(n-j+1)\right] \tag{31}$$

$$\gamma\left[\sum_{j=3}^{n-1}(j-1)(n-j+1)\right] \;=\; (n-2)*2-1*(n-1) \tag{32}$$

From the above we get

$$\gamma \;=\; \frac{(n-2)*2-1*(n-1)}{[\sum_{j=3}^{n-1}(j-1)(n-j+1)]} \;=\; \frac{3*(n-3)-1*(n-1)}{2*(n-2)+3*(n-3)+4*(n-4)+...+(n-2)*2} \tag{33}$$

This above equation is the same with $k = (n-2)$ in Eq.(4).