International Journal of Innovative Computing, Information and Control
Volume **4**, Number **8**, August **2008**

ICIC International ©2008 ISSN 1349-4198
pp. 2033–2043

PARAMETRIC ANALYSIS OF BI-CRITERION SINGLE MACHINE SCHEDULING WITH A LEARNING EFFECT

Pei Chann Chang[1], Shih Hsin Chen[2] and V. Mani[3]

[1]Department of Information Management
Yuan Ze University
135, Yuan-Dong Road, Tao-Yuan, 32026, Taiwan
iepchang@saturn.yzu.edu

[2]Department of Electronic Commerce Management
Nanhua University
32, Chungkeng, Dalin, Chiayi 62248, Taiwan

[3]Department of Aerospace Engineering
Indian Institute of Science
Bangalore 560-012, India

Received June 2007; revised December 2007

Abstract. *In a manufacturing system workers are involved in doing the same job or activity repeatedly. Hence, the workers start learning more about the job or activity. Because of the learning, the time to complete the job or activity starts decreasing, which is known as "learning effect." In this paper, we present a parametric analysis of bi-criterion single machine scheduling problem of n jobs with a learning effect. The two objectives considered are the total completion time (TC) and total absolute differences in completion times (TADC). The objective is to find a sequence of jobs that minimizes a linear combination of total completion time and total absolute differences in completion times; i.e.,* $\delta * TC + (1-\delta) * TADC$ $(0 \leq \delta \leq 1)$*. In an earlier study, this bi-criterion problem with a learning effect is formulated as an assignment problem and the optimal sequence is obtained, for a given value of* δ*. The computational complexity for solving an assignment problem is* $O(n^3)$*. In our study, the learning effect is included in the positional penalties/weights, and hence the simple matching procedure given in an another earlier study is used to obtain the optimal sequence. The complexity of the matching procedure is O(n log n). We show that the optimal sequence, depends on the value of* δ *and the learning index (*α*). In this paper, a parametric analysis of* δ*, for a given learning index (*α*) is presented to show the range of* δ *in which a sequence is optimal. We also present a method of obtaining the set of optimal sequences. A parametric analysis of* α *for a given* δ *is also presented. Numerical examples are presented for ease of understanding the methodology.*

Keywords: Single machine scheduling, Bi-criterion problem, Learning effect, Parametric analysis

1. **Introduction.** During the past fifty years, single machine scheduling problem has been studied by many researchers. A good introduction to sequencing and scheduling is given in [3], and also various issues related to single machine scheduling is presented. In conventional manufacturing systems, the processing time of a job is assumed to be a constant. When the workers are repeating the same job again and again, they start learning about the job. Because of this "learning effect," the processing time of a job is not a constant. This "learning effect" is first presented in [16] and is a well-known concept in management science literature. A survey of the learning effect observed in many practical situations is given in [18].

The learning effect in the context of single machine scheduling was first studied in [4]. In that study [4], two objectives minimal deviation from a common due date and minimum flow time are considered separately. For the solution of any of the objectives an assignment problem formulation is presented. The solution is obtained by solving the assignment problem. The assignment problem formulation given in [4] was continued with the objective of due-date assignment problem, simultaneous minimization of total completion time and variation of completion times in [11]. The scheduling problem with a learning effect on parallel identical machines is presented in [12]. The assignment problem formulation used in [4, 11] is polynomial solvable. It is known that the computational complexity of solving an assignment problem is $O(n^3)$. A bi-criterion single machine scheduling problem with a learning effect is considered in [9]. The two objectives considered in that study [9] are the total completion time and the maximum tardiness. In [17] some heuristic algorithms are presented for maximum lateness scheduling problem with a learning effect. In [10], a two machine flowshop with a learning effect is considered and their objective is minimization of total completion time. A branch and bound technique is presented for the solution and a heuristic algorithm is also presented in [10] to improve the efficiency of the branch and bound technique. Some important studies considering the learning effect are: [15, 13, 5, 1, 8].

In this paper, the learning effect is included as given in [4]. The processing time of a job depends on its position in the sequence and is given as follows:

$$p_{jl} \;=\; p_j l^{\alpha} \tag{1}$$

In the above equation, p_j is the normal processing time of job j, and p_{jl} is the processing time of job j if it is in position l of the sequence, and α is the learning index and $\alpha < 0$. According to equation (1), we see that $p_{j1} > p_{j2} > p_{j3}... > p_{jn}$. For example, if $p_j = 3$ and $\alpha = -0.515$, then $p_{j1} = 3$, $p_{j2} = 2.0994$, $p_{j3} = 1.7037$, $p_{j4} = 1.4691$, $p_{j5} = 1.3095$, and so on.

Motivation and contributions of this paper: Our motivation is to present a parametric analysis of bi-criterion single machine scheduling problem of n jobs with a learning effect. The two objectives considered are the total completion time (TC) and total absolute differences in completion times ($TADC$). The objective is to find a sequence that minimizes a linear combination of total completion time and total absolute differences in completion times; i.e., $\delta * TC \;+\; (1 - \delta) * TADC$. In an earlier study [11], this problem with a learning effect is formulated as an assignment problem and the optimal sequence is obtained, for a given value of δ and α. We show that the optimal sequence, depends on the value of δ and the learning index (α). Our study is motivated to conduct a parametric analysis of δ for a given α. We also present a parametric analysis of α for a given δ. The computational complexity for solving an assignment problem given in [11] is $O(n^3)$. In our method, the learning effect is included in the positional penalties/weights, and hence the simple matching procedure given in [2] is used to obtain the optimal sequence. The complexity of the matching procedure is $O(n \; log \; n)$. A parametric analysis of δ, for a given learning index (α) is presented to show the range of δ in which a sequence is optimal. We also present a method of obtaining the set of optimal sequences.

2. **Problem Formulation and Preliminary Analysis.**

We consider the single machine scheduling problem with a learning effect. A set of n independent jobs to be processed on a continuously available single machine. The machine can process only one job at a time and job splitting and inserting idle times are not permitted. Each job has a normal processing time p_j ($j = 1, 2, ..., n$) if they are at the first position in the sequence. The sequence is the order in which the jobs are processed in the machine. The jobs are numbered according to shortest normal processing time rule, i.e., $p_1 \leq p_2 \leq ... \leq p_n$. The two objectives considered are the total completion time (TC) and total absolute differences in completion times ($TADC$). TC and $TADC$ for a given sequence σ are

$$\begin{aligned} TC &= \sum_{j=1}^{n} C_j \\ TADC &= \sum_{i=1}^{n}\sum_{j=i}^{n} |C_i - C_j| \end{aligned} \tag{2}$$

where C_j is the completion time of job j in the given sequence σ. The objective is to find the sequence that minimizes the linear combination of both the objectives and is:

$$f(\sigma) \ = \delta TC \ + \ (1-\delta)TADC \ and \ 0 \leq \delta \leq 1. \tag{3}$$

Without learning effect $\alpha = 0$: In an earlier study [2], when there is no learning effect, this bi-criterion problem is solved using a simple matching procedure. The matching procedure is based on a well-known result in linear algebra that the sum of pairwise product of two vectors is minimized by arranging the elements of one vector in non-increasing order and the elements of other vector in non-decreasing order [2]. This we call as matching procedure in this paper. The computational complexity of this matching procedure is $O(n \log n)$. This matching procedure is used for single machine scheduling problems in [14, 7]. The objective in the bi-criterion problem without a learning effect is given as

$$f(\sigma) = \delta * \sum_{r=1}^{n}(n-r+1)p_{[r]} \ + \ (1-\delta) * \sum_{r=1}^{n}(r-1)(n-r+1)p_{[r]} \tag{4}$$

$$= \sum_{r=1}^{n} w_r^c p_{[r]} \tag{5}$$

In this problem, the objective is to obtain the schedule (i.e., the order in which the jobs are processed), and so this problem is handled by denoting any schedule by a permutation of job indices given as $[1], [2], ..., [n]$. Hence, in the above equation, $p_{[r]}$ is the processing time of job in position r. In that study [2], the optimal sequence for a given δ is obtained as follows:

- Obtain the positional weights for all the n positions.
- The positional weights for the objective TC is $w_r^1 = (n-r+1)$.
- The positional weights for the objective $TADC$ are $w_r^2 = (r-1)*(n-r+1)$, $r = 1, 2, ..., n$.
- Obtain the combined positional weight (w_r^c) given as

$$w_r^c \ = \ (2\delta - 1)(n+1) + r[2 - 3\delta + n(1-\delta)] - r^2(1-\delta) \tag{6}$$

- Obtain the optimal sequence (σ^*) by matching the position weights in descending order with jobs in ascending order of their normal processing times.

We note that the w_r^c is independent of $p_{[r]}$ and hence the matching algorithm is able to obtain the optimal schedule.

With a learning effect $\alpha < 0$: Now, we will include the learning effect in our analysis. The learning effect is only dependent on the position. So we include the learning effect in the positional weights. So the positional weights with a learning effect for the objective

TC and $TADC$ are:

$$w_r^{1,\alpha} = (n-r+1)*r^\alpha \tag{7}$$
$$w_r^{2,\alpha} = (r-1)(n-r+1)*r^\alpha \tag{8}$$

The combined positional weights with a learning effect is

$$w_r^{c,\alpha} = \{(2\delta-1)(n+1)+r[2-3\delta+n(1-\delta)]-r^2(1-\delta)\}*r^\alpha \tag{9}$$

We know that $w_r^{c,\alpha}$ is also independent of $p_{[j]}$. So we can use the matching algorithm and obtain the optimal schedule for a given value of δ and α. The learning index is α and $\alpha \leq 0$. We can also see if $\alpha = 0$ equation (9) reduces to equation (6).

Example 2.1. *Let us consider the 7 job problem given in* [2]. *The normal processing time of these jobs are* $p_1 = 2$, $p_2 = 3$, $p_3 = 6$, $p_4 = 9$, $p_5 = 21$, $p_6 = 65$, *and* $p_7 = 82$. *Let the value of* $\delta = 0.5$.

For this 7 job problem, the value of TC and $TADC$ for a given schedule is

$$TC = 7*p_{[1]}+6*p_{[2]}+5*p_{[3]}+4*p_{[4]}+3*p_{[5]}+2*p_{[6]}+1*p_{[7]} \tag{10}$$
$$TADC = 6*p_{[2]}+10*p_{[3]}+12*p_{[4]}+12*p_{[5]}+10*p_{[6]}+6*p_{[7]} \tag{11}$$

For a given learning rate $\alpha = -0.322$ the positional weights obtained from equation (9) are $w_1^{c,\alpha} = 3.5$, $w_2^{c,\alpha} = 4.7798$, $w_3^{c,\alpha} = 5.2654$, $w_4^{c,\alpha} = 5.1195$, $w_5^{c,\alpha} = 4.4668$, $w_6^{c,\alpha} = 3.3697$, and $w_7^{c,\alpha} = 1.8705$.

In the matching algorithm, the optimal sequence (σ^*) is obtained by matching the position weights in descending order with jobs in ascending order of their normal processing times. Using the matching algorithm, we obtain the sequence {5 3 1 2 4 6 7} as the optimal sequence. The same sequence is obtained by solving an assignment problem in [11]. The positional weights and the optimal sequence obtained are shown in Table 1

TABLE 1. Bi-criterion problem $\alpha = -0.322$ and $\delta = 0.5$

Position-r	1	2	3	4	5	6	7
$w_r^{c,\alpha}$	3.50000	4.7798	5.2654	5.1195	4.4668	3.3697	1.8705
Sequence*	**5**	**3**	**1**	**2**	**4**	**6**	**7**

In the above example, we are able to obtain the optimal sequence for a given value of δ and the learning index α using the matching algorithm. The complexity of this matching algorithm is $O(n \log n)$.

3. **Parametric Analysis of δ.** A parametric analysis for this bi-criterion problem, when there is no learning ($\alpha = 0$) is presented in [2]. We can easily obtain the optimal sequence using the matching algorithm, for a given value of δ and α. For a given value of α, finding the appropriate δ is not a trivial problem. This fact has been brought out in [2]. When $\alpha = 0$, [2] has presented a number of interesting results in his parametric analysis. In that study [2], first a complete set of optimal schedules (CSOS) is obtained and then obtain minimum set of optimal solutions (MSOS). MSOS is a subset of CSOS. The MSOS has the property that for any δ from $(0, 1)$, at least one and at most two of the optimal solutions to this problem (3) are members of this set CSOS. The cardinality of MSOS is n the number of jobs. An $O(n^2)$ algorithm for determination of this set MSOS is presented in [2].

We will consider the same 4 job problem given in [2], with a learning effect. The normal processing time of the jobs are $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, and $p_4 = 4$. For this 4 job problem, the value of TC and $TADC$ for a given schedule is

$$TC = 4 * p_{[1]} + 3 * p_{[2]} + 2 * p_{[3]} + 1 * p_{[4]} \quad (12)$$

$$TADC = 3 * p_{[2]} + 4 * p_{[3]} + 3 * p_{[4]} \quad (13)$$

The positional weights $w_r^{c,\alpha}$ given by (9), is a linear function of δ for a given learning index α. The values of positional weights for the 4 positions are:

- $w_1^{c,\alpha} = 4 * \delta * 1^{\alpha}$
- $w_2^{c,\alpha} = 3 * 2^{\alpha}$
- $w_3^{c,\alpha} = (4 - 2 * \delta) * 3^{\alpha}$
- $w_4^{c,\alpha} = (3 - 2 * \delta) * 4^{\alpha}$

Because of the learning effect ($\alpha \neq 0$) for any value of δ from $(0, 1)$, we obtain at least one and at most two optimal solutions. Hence, when $\alpha \neq 0$, the complete set of optimal schedules (CSOS) is the same as minimum set of optimal solutions (MSOS). Also, when $\alpha \neq 0$, the cardinality of MSOS is not equal to n. For the case with a learning effect ($\alpha \neq 0$), we present an $O(n^2)$ algorithm for determination of this set MSOS.

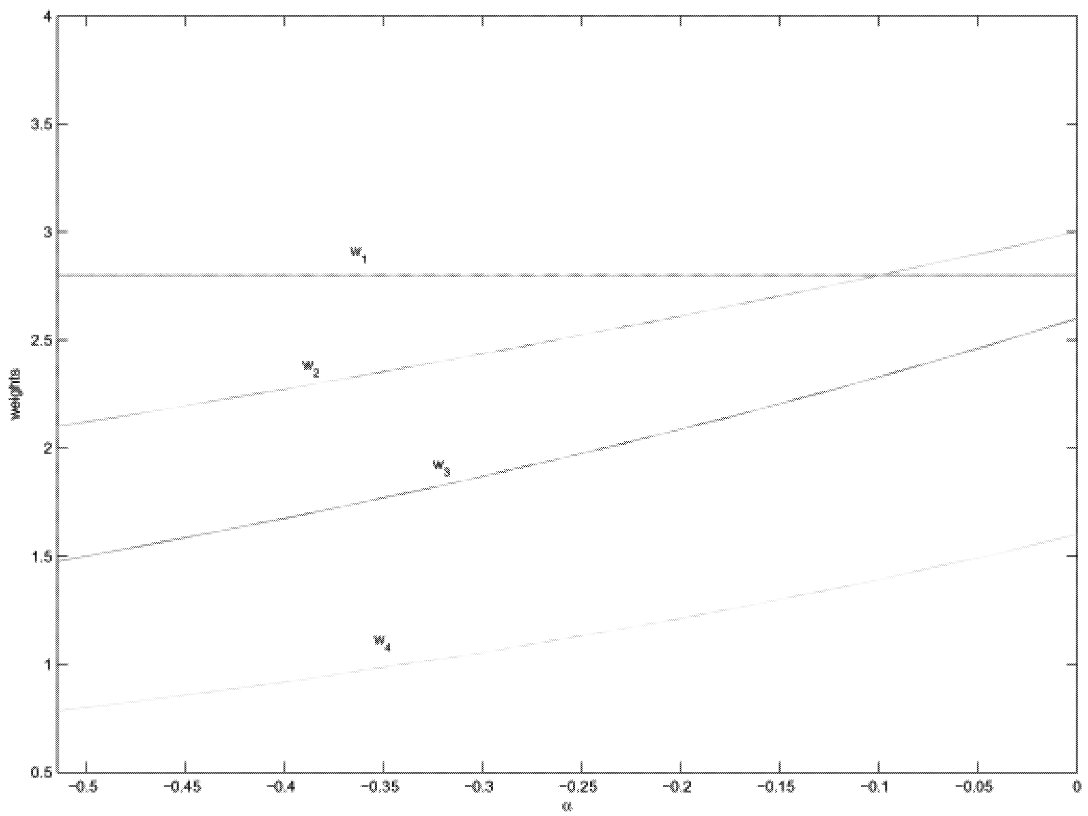

FIGURE 1. Positional weights as a function of δ for $\alpha = -0.152$

We notice that the positional weights $w_1^{c,\alpha}$, $w_2^{c,\alpha}$, $w_3^{c,\alpha}$, and $w_4^{c,\alpha}$ are linear functions of δ for a given learning rate (α). In Figure 1, we plot these positional weights as a function of δ, for the given value of learning rate ($\alpha = -0.152$) which corresponds to 90% learning. The range of δ in which a sequence is optimal is obtained using the point at which two lines of $w_r^{c,\alpha}$ ($r = 1, 2, 3, 4$) intersect in Figure 1. For this 4 job problem there are four points of intersection for δ in (0,1). These intersection points can be obtained by equating the corresponding positional weights. There are six points of intersection denoted as δ_1 to δ_6, and are:

- Intersection of $w_1^{c,\alpha}$ and $w_2^{c,\alpha}$ is δ_1
- Intersection of $w_1^{c,\alpha}$ and $w_3^{c,\alpha}$ is δ_2
- Intersection of $w_1^{c,\alpha}$ and $w_4^{c,\alpha}$ is δ_3
- Intersection of $w_2^{c,\alpha}$ and $w_3^{c,\alpha}$ is δ_4
- Intersection of $w_2^{c,\alpha}$ and $w_4^{c,\alpha}$ is δ_5
- Intersection of $w_3^{c,\alpha}$ and $w_4^{c,\alpha}$ is δ_6

The values of δ_1 to δ_6 are given by

$$\delta_1 = \frac{3 * 2^\alpha}{4 * 1^\alpha} \tag{14}$$

$$\delta_2 = \frac{4 * 3^\alpha}{(4 * 1^\alpha + 2 * 3^\alpha)} \tag{15}$$

$$\delta_3 = \frac{3 * 4^\alpha}{(4 * 1^\alpha + 2 * 4^\alpha)} \tag{16}$$

$$\delta_4 = \frac{(4 * 3^\alpha - 3 * 2^\alpha)}{(2 * 3^\alpha)} \tag{17}$$

$$\delta_5 = \frac{(3 * 4^\alpha - 3 * 2^\alpha)}{(2 * 4^\alpha)} \tag{18}$$

$$\delta_6 = \frac{(4 * 3^\alpha - 3 * 4^\alpha)}{(2 * 3^\alpha - 2 * 4^\alpha)} \tag{19}$$

For the learning index $\alpha = -0.152$, the above values are: $\delta_1 = 0.67500$, $\delta_2 = 0.594622$, $\delta_3 = 0.432386$, $\delta_4 = 0.404645$, $\delta_5 = 3.166666$, and $\delta_6 = 13.186228$. We are interested in finding the δ value in (0,1), so we discard δ_5 and δ_6. These points of intersection are also shown in Figure 1.

For a given value of learning index ($\alpha = -0.152$), the range of δ and the optimal sequence in that range are shown in Table 2. From our results, we can easily see that the optimal sequence depends on the value of δ. Also, we can see for any given sequence, the total penalty decreases because of the learning effect. We have also considered the learning index $\alpha = -0.322$ and $\alpha = -0.515$. The results are shown in Tables 3, 4.

TABLE 2. Range of δ and the optimal sequence for $\alpha = -0.152$

Range of δ	Optimal Sequence
0.0 to 0.404645	$\{4, 2, 1, 3\}$
0.404645 to 0.432386	$\{4, 1, 2, 3\}$
0.432386 to 0.594622	$\{3, 1, 2, 4\}$
0.594622 to 0.675000	$\{2, 1, 3, 4\}$
0.675000 to 1.0000	$\{1, 2, 3, 4\}$

3.1. **Parametric analysis of learning index (α).** Now, we will consider the value of δ is known, and we will find the range of α in which a sequence is optimal. The positional weights $w_r^{c,\alpha}$ given by (9), is a non-linear function of α (except for $w_1^{c,\alpha}$) for a given value of δ. We will consider the same 4 job problem given in [2]. In Figure 2, we

TABLE 3. Range of δ and the optimal sequence for $\alpha = -0.322$

Range of δ	Optimal Sequence
0.0 to 0.143327	$\{4, 2, 1, 3\}$
0.143327 to 0.363609	$\{4, 1, 2, 3\}$
0.363609 to 0.519641	$\{3, 1, 2, 4\}$
0.519641 to 0.599970	$\{2, 1, 3, 4\}$
0.599970 to 1.0000	$\{1, 2, 3, 4\}$

TABLE 4. Range of δ and the optimal sequence for $\alpha = -0.515$

Range of δ	Optimal Sequence
0.0 to 0.048919	$\{4, 2, 1, 3\}$
0.048919 to 0.295056	$\{4, 1, 2, 3\}$
0.295056 to 0.4423154	$\{3, 1, 2, 4\}$
0.4423154 to 0.5248446	$\{2, 1, 3, 4\}$
0.5248446 to 1.0000	$\{1, 2, 3, 4\}$

plot these positional weights as a function of α, for the given value of ($\delta = 0.7$). The range of α in which a sequence is optimal is obtained using the point at which two of $w_r^{c,\alpha}$ ($r = 1, 2, 3, 4$) intersect in Figure 2. These intersection points can be obtained by equating the corresponding positional weights.

When $\alpha = 0$, for $\delta = 0.7$, the optimal sequence is $\{2\ 1\ 3\ 4\}$. The range of α in which the sequence $\{2\ 1\ 3\ 4\}$ is optimal is obtained by equating $w_1^{c,\alpha}$ and $w_2^{c,\alpha}$. It is obtained as

$$\begin{aligned} \alpha &= log(4 * \delta/3)/log(2) \\ &= -0.0995356 \end{aligned} \tag{20}$$

Hence, for $\delta = 0.7$, the sequence $\{2\ 1\ 3\ 4\}$ is optimal, when α is in the range of -0.0995356 *to* 0. In a similar manner, we can find the range of α for other optimal sequences for a given value of δ.

We have to solve a non-linear equation to obtain the range of α, for a given δ. But, we have to solve a linear equation to obtain the range of δ, for a given α.

Generalization: We will now generalize our methodology to n jobs with a given learning index α. Compute the weights for these seven jobs $w_r^{c,\alpha}$ for $r = 1, 2, ..., 7$. There are $n * (n-1)$ intersection points between the positional weights $w_r^{c,\alpha}$. This also means that there are $n * (n-1)$ distinct values of δ are possible. These $n * (n-1)$ values of δ can be computed by equating the appropriate positional weights. When the value of δ is less than zero or greater than one, discard those values. Let us assume that there are L distinct values of δ in (0,1). Arrange these δ values in decreasing order. Let the values of δ be:$\{\delta_1, \delta_2, ..., \delta_L\}$.

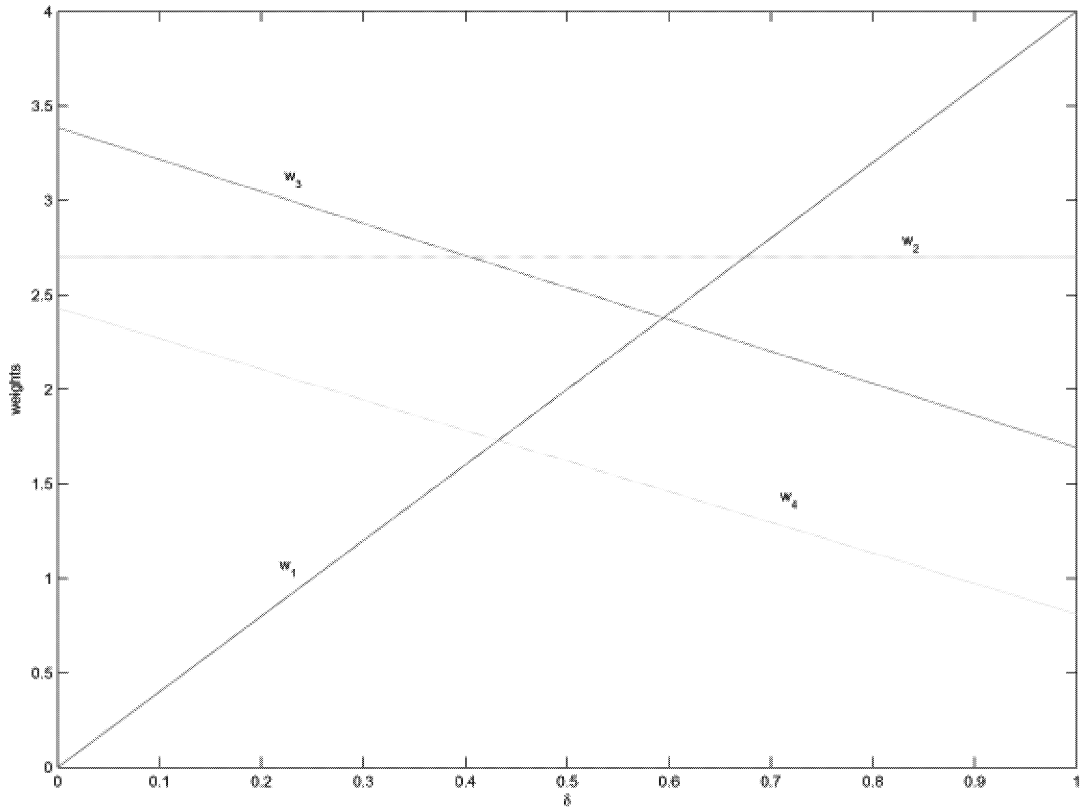

FIGURE 2. Positional weights as a function of α for $\delta = 0.7$

Choose a δ value in between 1 and δ_1, and obtain the sequence using the matching method. This sequence is the optimal sequence for δ in $(0,\delta_1)$. Choose a δ value in between δ_1 and δ_2, and obtain the sequence using the matching method. This sequence is the optimal sequence for δ in (δ_1,δ_2). Continue this process till the chosen value of δ is in between δ_{L-1} and δ_L. After this choose a value of δ in between δ_L and 0, and obtain the sequence using the matching method. This sequence is the optimal sequence for δ in $(\delta_L, 0)$. In general we will have $L + 1$, optimal sequences. Note that the value of L depends on the learning index α.

We now explain the above generalization by considering the 7 job problem given in [2], with a learning effect. The value of learning index is $\alpha = -0.152$. The normal processing times are $p_1 = 2$, $p_2 = 3$, $p_3 = 6$, $p_4 = 9$, $p_5 = 21$, $p_6 = 65$, and $p_7 = 82$. As mentioned earlier, the positional weights $w_r^{c,\alpha}$ given by (9), is a linear function of δ for a given learning index $\alpha = -0.152$. The values of positional weights for the 7 positions are:

- $w_1^{c,\alpha} = 7 * \delta * 1^{\alpha}$
- $w_2^{c,\alpha} = 6 * 2^{\alpha}$
- $w_3^{c,\alpha} = (10 - 5 * \delta) * 3^{\alpha}$
- $w_4^{c,\alpha} = (12 - 8 * \delta) * 4^{\alpha}$
- $w_5^{c,\alpha} = (12 - 9 * \delta) * 5^{\alpha}$
- $w_6^{c,\alpha} = (10 - 8 * \delta) * 6^{\alpha}$
- $w_7^{c,\alpha} = (6 - 5 * \delta) * 7^{\alpha}$

Following the generalization, there are $n * (n - 1)$ intersection points between the positional weights $w_r^{c,\alpha}$. We are interested in δ values in the interval $(0, 1)$. The δ values obtained in this interval are:

- $w_1^{c,\alpha}$ and $w_2^{c,\alpha}$ intersect at point $\delta = 0.77143$.

- $w_1^{c,\alpha}$ and $w_3^{c,\alpha}$ intersect at point $\delta = 0.75345$
- $w_1^{c,\alpha}$ and $w_4^{c,\alpha}$ intersect at point $\delta = 0.72017$
- $w_1^{c,\alpha}$ and $w_5^{c,\alpha}$ intersect at point $\delta = 0.66889$
- $w_1^{c,\alpha}$ and $w_6^{c,\alpha}$ intersect at point $\delta = 0.58169$
- $w_1^{c,\alpha}$ and $w_7^{c,\alpha}$ intersect at point $\delta = 0.41640$
- $w_2^{c,\alpha}$ and $w_3^{c,\alpha}$ intersect at point $\delta = 0.72372$
- $w_2^{c,\alpha}$ and $w_4^{c,\alpha}$ intersect at point $\delta = 0.66667$
- $w_2^{c,\alpha}$ and $w_5^{c,\alpha}$ intersect at point $\delta = 0.56704$
- $w_2^{c,\alpha}$ and $w_6^{c,\alpha}$ intersect at point $\delta = 0.36369$
- $w_3^{c,\alpha}$ and $w_4^{c,\alpha}$ intersect at point $\delta = 0.55934$
- $w_3^{c,\alpha}$ and $w_5^{c,\alpha}$ intersect at point $\delta = 0.33162$

We see that there are 12 distinct values of δ in (0,1). We arrange these δ values in decreasing order. We choose a δ value (in between two δ values given above) and obtain the sequence using the matching method. In this manner, we obtain 13 optimal sequences. The optimal sequences and the range of δ are shown in Table 5.

TABLE 5. Range of δ and the optimal sequence for $\alpha = -0.152$

Range of δ	Optimal Sequence
0.0 to 0.3316	$\{7,5,3,1,2,4,6\}$
0.3316 to 0.3636	$\{7,5,2,1,3,4,6\}$
0.3636 to 0.4164	$\{7,4,2,1,3,5,6\}$
0.4164 to 0.5593	$\{6,4,2,1,3,5,7\}$
0.5593 to 0.5670	$\{6,4,1,2,3,5,7\}$
0.5670 to 0.5816	$\{6,3,1,2,4,5,7\}$
0.5816 to 0.6666	$\{5,3,1,2,4,6,7\}$
0.6666 to 0.6688	$\{5,2,1,3,4,6,7\}$
0.6688 to 0.7202	$\{4,2,1,3,5,6,7\}$
0.7202 to 0.7237	$\{3,1,2,4,5,6,7\}$
0.7237 to 0.7534	$\{3,2,1,4,5,6,7\}$
0.7534 to 0.7714	$\{2,1,3,4,5,6,7\}$
0.7714 to 1.0000	$\{1,2,3,4,5,6,7\}$

4. **Discussions.** When the learning effect is not included ($\alpha = 0$), it is shown in [2] that the minimum set of optimal solutions (MSOS) is n. We have shown that when $\alpha \neq 0$ the minimum set of optimal solutions (MSOS) is not equal to n. It is shown in our study that the optimal sequence depends on the value of both α and δ. We have shown the range of δ and the optimal sequence in that range, for a given learning index (α). We have also shown a parametric analysis of α for a given δ.

In an earlier study [11], this bi-criterion problem with a learning effect is formulated as an assignment problem and the optimal sequence is obtained, for a given value of δ. The computational complexity for solving an assignment problem is $O(n^3)$. The complexity of the matching procedure presented in our study is O(n log n), which is an advantage.

In the parametric analysis presented, the bi-criterion problem is converted to a single objective problem by a linear combination of the two objectives as a weighted sum. The weights are δ and $(1-\delta)$. The set of non-inferior solution points can be obtained by varying the weights. We have another variable the learning index (α). So in this method, for a given learning index (α), we can obtain the optimal sequence and also the range of the weight (δ). Also, we can obtain the range of α for which a sequence is optimal for a given δ. It is known that this parametric analysis can be used for any bi-criterion optimization problem. Other methods of solving bi-criterion problem is adjacent efficient point method and the adjacent efficient basis method. Our interest is to obtain the range of δ in which a sequence is optimal and so we have used the parametric analysis.

In general, the methodology of multi-objective optimization methods can be used in many application areas. A multi-objective programming is proposed for the imprecise data envelopment analysis in [19]. A multi-criteria humanoid robot analysis is presented in [6]. In the study [6], if only two objectives are considered then it is possible to conduct a parametric analysis in their evolutionary approach.

5. **Conclusions.** In this paper, we consider the bi-criterion single machine scheduling problem of n jobs with a learning effect. The two objectives considered are, the total completion time (TC) and the total absolute differences in completion times ($TADC$). The objective is to find a sequence that minimizes a linear combination of total completion time and total absolute differences in completion times; i.e., $\delta * TC + (1-\delta) * TADC$. In an earlier study [11], this problem with a learning effect is formulated as an assignment problem and the optimal sequence is obtained, for a given value of δ. The computational complexity for solving an assignment problem is $O(n^3)$. In our study, we have included the learning effect in the positional penalties/weights, and hence the simple matching procedure given in an earlier study is used to obtain the optimal sequence. The complexity of the matching procedure is O(n log n). We have shown that the optimal sequence depends on the value of δ and the learning index (α). A parametric analysis of δ, for a given learning index is presented to show the range of δ in which a sequence is optimal. A parametric analysis of α for a given δ is presented. We also presented a method of obtaining the set of optimal sequences.

Acknowledgements: The authors would like to thank the reviewers for their suggestions and the editor for his encouragement.

REFERENCES

[1] A. Bachman and A. Janiak, Scheduling jobs with position-dependent processing times, *Journal of the Operational Research Society*, vol.55, no.3, pp.257-264, 2004.
[2] U. Bagchi, Simultaneous minimization of mean and variation of flow time and waiting time in single machine systems, *Operations Research*, vol.37, no.1, pp.118-125, 1989.
[3] K. R. Baker, *Introduction to Sequencing and Scheduling*, John Wiley & Sons, New York, 1974.
[4] D. Biskup, Single-machine scheduling with learning considerations, *European Journal of Operational Research*, vol.115, no.1, pp.173-178, 1999.
[5] D. Biskup and D. Simons, Common due date scheduling with autonomous and induced learning, *European Journal of Operational Research*, vol.159, no.3, pp.606-616, 2004.
[6] C. Capi, M. Yokota and K. Mitobe, Optimal multi-criterion humanoid robot gait synthesis - An evolutionary approach, *International Journal of Innovative Computing, Information and Control*, vol.2, no.6, pp.1249-1258, 2006.
[7] T. C. E. Cheng, Optimal single machine sequencing and assignment of common due-dates, *Computers and Industrial Engineering*, vol.22, no.2, pp.115-120, 1992.
[8] W. H. Kuo and D. L. Yang, Minimizing the makespan in a single machine scheduling problem with a time based learning effect, *Information Processing Letters*, vol.97, no.2, pp.64-67, 2006.
[9] W. C. Lee, C. C. Wu and H. J. Sung, A bi-criterion single-machine scheduling problem with learning considerations, *Acta Informatica*, vol.40, no.4, pp.303-315, 2004.
[10] W. C. Lee and C. C. Wu, Minimizing total completion time in a two-machine flowshop with a learning effect, *International Journal of Production Economics*, vol.88, no.1, pp.85-93, 2004.

[11] G. Mosheiov, Scheduling problems with a learning effect, *European Journal of Operational Research*, vol.132, no.3, pp.687-693, 2001.

[12] G. Mosheiov, Parallel machine scheduling with a learning effect, *Journal of the Operational Research Society*, vol.52, no.10, pp.1165-1169, 2001.

[13] G. Mosheiov and J. B. Sidney, Note on scheduling with general learning curves to minimize the number of tardy jobs, *Journal of the Operational Research Society*, vol.56, no.1, pp.110-112, 2005.

[14] S. S. Panwalkar, M. L. Smith and A. Seidmann, Common due-date assignment to minimize total penalty for the one machine scheduling problem, *Operations Research*, vol.30, no.2, pp.391-399, 1982.

[15] J. B. Wang and Z. Q. Xia, Flowshop scheduling with a learning effect, *Journal of the Operational Research Society*, vol.56, no.11, pp.1325-1330, 2005.

[16] T. P. Wright, Factors affecting the cost of airplanes, *Journal of Aeronautical Science*, vol.3, no.2, pp.122-128, 1936.

[17] C. C. Wu, W. C. Lee and T. Chen, Heuristic algorithms for solving the maximum lateness scheduling problem with considerations, *Computers and Industrial Engineering*, vol.52, no.1, pp.124-132, 2007.

[18] L. E. Yelle, The learning Curve: Historical review and comprehensive survey, *Decision Science*, vol.10, no.2, pp.302-328, 1979.

[19] J. R. Yu, IDEA by multi objective programming, *International Journal of Innovative Computing, Information and Control*, vol.3, no.1, pp.21-29, 2007.